Math 347: Homework 1
Due on: Sep. 7, 2018

General advice for the exercises: You should make sure that you are able to do the ones marked by a (*). Most of the exercises refer to abstract sets and ask you to prove properties about them. Write down examples to get an understanding of what is happening.

1. (*) Consider the sets: $A = \{1, 3, 5\}$, $B = \{3, 4, 6\}$, $C = \{5\}$, and $D = \{1, 3\}$, and assume that the universe is $\{1, 2, 3, 4, 5, 6\}$.
   - (i) Which sets are subsets of the others?
   - (ii) For which sets S do we have $1 \in S$, and $1 \notin S$?
   - (iii) Which sets are not subsets of each other?
   - (iv) What is $A \backslash B$, $A \backslash C$?
   - (v) Which sets are disjoint?
   - (vi) What is A^c, B^c, C^c and D^c?[1]
   - (vii) Find $A \cup B$ and $A \cap B$.
2. (*) Prove the following properties about sets:
   - (i) $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$;
   - (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
   - (iii) $A \cap B = A$ if and only if $A \subseteq B$.
3. (*) What is the relation between $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$? Is one set contained in the other? What can you say if $A \subseteq B$, what if $B \subseteq A$?
4. Wikipedia defines a *lattice* to be the following[2]. A set L equipped with two operations $\wedge$ and $\vee$: satisfying the following list of axioms:
   - (C1) $A \vee B = B \vee A$;
   - (C2) $A \wedge B = B \wedge A$;
   - (A1) $A \vee (B \vee C) = (A \vee B) \vee C$;
   - (A2) $A \wedge (B \wedge C) = (A \wedge B) \wedge C$;
   - (Ab1) $A \vee (A \wedge B) = A$;
   - (Ab2) $A \wedge (A \vee B) = A$.

   For any set A consider its power set $P(A)$. Prove that there are $\vee$ and $\wedge$ operations defined on $P(A)$ that make it into a lattice. The wikipedia page has a lot more information, can you find another structure there such that $P(A)$ with some more data is an example of?

---

[1] See Definition 1.18 in the book for what this means.

[2] See section General lattice in https://en.wikipedia.org/wiki/Lattice-(order).

5. (*) Given a function $f : A \to B$ and a subset $S \subseteq B$, $f^{-1}(S)$ is another notation for $I_f(S)$ as defined in Exercise 6 from the Worksheet - Lesson 2, or in the textbook. Prove that for any subset $T \subseteq A$ one has

$$T \subseteq f^{-1}(f(T)).$$

Find an example where the equality doesn't hold.

6. (*) Let A and B be sets, recall that the cartesian product $A \times B$ is the following set

$$\{(a, b) \mid a \in A, b \in B\}.$$

Notice that one has two functions:

$$p_1 : A \times B \to A \quad \text{and} \quad p_2 : A \times B \to B.$$

given by

$$p_1((a, b)) = a \quad \text{and} \quad p_2((a, b)) = b.$$

What is $p_1^{-1}(A)$? For $S \subseteq B$, what is $p_2^{-1}(S)$?

7. Given a function $f : A \to B$ one can define a subset of $A \times B$ called the *graph of* f as follows

$$\Gamma_f = \{(a, b) \in A \times B \mid f(a) = b\}.$$

So one can think of the data of a function as a subset of $A \times B$. Determine if the following are true or false with justification.

   (i) Any subset $S \subseteq A \times B$ determines a function $f : A \to B$.

   (ii) A subset $S \subseteq A \times B$ determines a function if $S \cap p_2^{-1}(\{b\})$ has a single element for all $b \in B$.

   (iii) A subset $S \subseteq A \times B$ determines a function if $S \cap p_1^{-1}(\{a\})$ has a single element for all $a \in B$.

8. Let $A = \{(2n + 1)^3 \mid n \in \mathbb{Z}\}$ and $B = \{2n + 1 \mid n \in \mathbb{Z}\}$ be two sets.

   (i) (*) Prove that $A \subseteq B$.

   (ii) Suppose that we redefine A and B, by replacing $\mathbb{Z}$ by $\mathbb{R}$, i.e. $A = \{(2n + 1) \mid n \in \mathbb{R}\}$ and similarly for B. What is the relation between A and B? State and prove your answer.

9. Consider the following three sets:

$$A = \{(x, y) \in \mathbb{R}^2 \mid xy > 0\};$$
$$B = \{(x, y) \in \mathbb{R}^2 \quad y > |x|\};$$
$$C = \{(x, y) \in \mathbb{R}^2 \quad 0 < x < y\}.$$

Carefully prove that $A \cap B = C$.